

# The issue of zeroth law for Killing horizons in Lanczos-Lovelock gravity

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We study the zeroth law for Killing horizons in Lanczos-Lovelock gravity. We show that the surface gravity of a general Killing horizon in Lanczos-Lovelock gravity (except for general relativity) may not be constant even when the matter source satisfies dominant energy condition.

General relativity (GR), being quantum mechanically non-renormalizable, may make sense as a Wilsonian effective theory working perturbatively in powers of the dimensionless small parameter  $G(\text{Energy})^{D-2}$ , where  $G$  is the  $D$ -dimensional Newton's constant. Then the Einstein-Hilbert Lagrangian is the lowest order term (other than the cosmological constant) in a derivative expansion of generally covariant actions for a metric theory, and the presence of higher curvature terms is presumably inevitable. In general, the specific form of these terms will depend on the detailed features of the quantum gravity model. Still, from a purely classical point of view, a natural modification of the Einstein-Hilbert action is to include terms preserving the diffeomorphism invariance and still leading to an equation of motion containing no more than second order time derivatives. Interestingly, this generalization is unique [1, 2] and goes by the name of Lanczos-Lovelock gravity. Lanczos-Lovelock gravity is free from perturbative ghosts [3] and leads to a well-defined initial value formalism [4]. The lowest order Lanczos-Lovelock correction term in space time dimensions  $D > 4$ , namely the Gauss-Bonnet term, also appears as a low energy  $\alpha'$  correction in case of heterotic string theory [3, 5]. Hence, it is interesting to pursue various classical and semi classical properties of Lanczos-Lovelock gravity. For example, the striking similarity of the laws of black hole mechanics with thermodynamics was first established in the case of general relativity [6] and a natural question is to ask whether this analogy is a peculiar property of GR or a robust feature of any generally covariant theory of gravity. Studying the properties of black holes in a general Lanczos-Lovelock theory may provide a partial answer to this important question.

The equilibrium state version of first law for black holes was established by Wald and collaborators [7, 8] for any arbitrary diffeomorphism invariant theory of gravity. The entropy of the black hole can be expressed as a local geometric quantity integrated over a space-like cross section of the horizon and is associated with the Noether charge of Killing isometry that generates the horizon.

It is also possible to write down a quasi stationary version of the second law for Lanczos-Lovelock gravity [9, 10] which proves that the entropy of black holes in Lanczos-Lovelock gravity monotonically increases for physical processes in which the horizon is perturbed by the accretion of positive energy matter and the black hole ultimately settles down to a stationary state.

On the other hand, the zeroth law of black hole mechanics which asserts the constancy of surface gravity has two independent versions. First, zeroth law can be established for stationary Killing horizons in GR, provided the matter obeys dominant energy condition [6]. The other version states that the surface gravity is constant on the horizon of a static or stationary-axisymmetric black hole with the  $t - \phi$  orthogonality property [11, 13]. The second version is entirely geometrical and independent of the field equation, whereas, the first version does not require the assumption of  $t - \phi$  orthogonality property, but is only valid for GR, i.e. when Einstein's equation with matter obeying dominant energy condition holds.

Motivated by the fact that both the first law and a quasi stationary second law hold true for Lanczos-Lovelock gravity, we study the zeroth law for a general Killing horizon in Lanczos-Lovelock gravity. We would like to see that if one uses Lanczos-Lovelock equation of motion and suitable energy condition, then whether it is possible to prove the constancy of surface gravity without any assumption of extra symmetry. In fact, in this paper, we provide a negative answer to this question and show that the constancy of surface gravity does not hold in general even when the matter source obeys dominant energy condition.

The paper is organized as follows: we first review the properties of Killing horizons. Next, we discuss the Lanczos-Lovelock theory and present the main result. Finally, we conclude with further discussions. We adopt the metric signature  $(-, +, +, +, \dots)$  and our sign conventions are same as those of [12].

Also, we note that in general, a Killing horizon is not necessarily an event horizon. But there is a version of rigidity theorem [11] which states that for a static black hole, the static Killing field must be normal to the horizon, whereas for a stationary-axisymmetric black hole with the  $t - \phi$  orthogonality property, there exists a Killing

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field which is normal to the event horizon. In case of GR, it is possible to show [14, 15] that the event horizon of a stationary black hole is also a Killing horizon with no assumptions of symmetries beyond stationarity. We are not aware of any such proof for Lanczos-Lovelock gravity.

In a  $D$ -dimensional space time, a Killing horizon (not necessarily an event horizon) is a null hyper-surface  $\mathcal{H}$  whose null generators are the orbits of a Killing field  $\xi^a = (\partial/\partial v)^a$ , which is null on the horizon. Then there exists a function, surface gravity, “ $\kappa$ ” of the Killing horizon which is defined by the equation,

$$\xi^a \nabla_a \xi^b = \kappa \xi^b. \quad (1)$$

For static black holes, it is possible to provide a physical interpretation of the surface gravity. In that case, surface gravity of the black hole horizon measures the force which must be exerted at infinity to hold an unit mass at horizon. In general case, the surface gravity is the measure of the failure of the Killing time to be the affine parameter along the horizon null generators. From the definition in Eq.(1), it is straightforward to show that the surface gravity is constant along a generator [6, 12], i.e.  $\xi^a \nabla_a \kappa = 0$ . In general, surface gravity may vary from one generator to the other. Note that, the definition of surface gravity requires the notion of stationarity. There is no notion of surface gravity for a general non stationary dynamical horizon.

The real significance of the surface gravity is realized when one consider quantum effects in a space time containing a black hole. The semi classical calculations by Hawking [16] showed that the black hole emits thermal radiation with a temperature (in units with  $G = c = k = 1$ ),

$$T = \frac{\hbar \kappa}{2\pi}. \quad (2)$$

Hawking’s result immediately shows the importance of the zeroth law of black hole mechanics as the zeroth law of black hole thermodynamics which states that the Hawking temperature is uniform everywhere on a stationary black hole horizon. This is reminiscent of the zeroth law of thermodynamics which states that the temperature is uniform everywhere in a system in thermal equilibrium.

Since the constancy of the surface gravity along a generator of the horizon follows directly from the definition in Eq.(1), we will only discuss the change of “ $\kappa$ ” from generator to generator. In order to proceed, we first construct a basis  $\{\xi^a, N^a, e_A^a\}$  on the Killing horizon where  $\xi^a$  is the Killing field,  $N^a$  is another null vector satisfying  $\xi^a N_a = -1$ ,  $e_A^a$ ,  $\{A = 2, \dots, D-1\}$  are the  $(D-2)$  space like vectors along the transverse directions and  $\xi^b \gamma_b^a = N^b \gamma_b^a = 0$ . Here  $\gamma_b^a$  is the induced metric of any space like slice of the horizon. We decompose the space time metric as  $g_{ab} = g_{ab}^\perp + \gamma_{ab}$ , where  $g_{ab}^\perp = -2\xi_{(a} N_{b)}$ , is the metric of the two dimensional space orthogonal to any horizon cross section. Also, for stationary space times with a Killing horizon, both the expansion and

shear vanish and using Raychaudhuri equation and the evolution equation for shear, we obtain [12, 17] that on the horizon,

$$R_{ab} \xi^a \xi^b = 0 \quad (3)$$

and

$$\xi^a \gamma_i^b \gamma_j^c \gamma_k^d R_{abcd} = 0. \quad (4)$$

We would like to emphasize that in order to derive these relationships, we have only used the fact that the horizon is a Killing horizon with zero expansion and shear without assuming any further symmetry.

Next, we would like to study how the surface gravity changes from one generator to the other. For that, we note that from Eq.(1), we can write  $\kappa = -\xi^a N^b \nabla_a \xi_b$  and then we obtain [6],

$$\gamma_b^a \nabla_a \kappa = -\xi^a R_{ac} \gamma_b^c. \quad (5)$$

So far, all the results are entirely geometrical and no use of the equation of motion is made. Now, if we further assume that the horizon is axisymmetric and possesses  $t - \phi$  orthogonality property, then it is possible to show [11, 13] that the RHS of Eq.(5) vanishes identically and the surface gravity is constant on the horizon independent of the field equation.

Note that, if we assume that the Killing horizon possesses a bifurcation surface, i.e. a  $(D-1)$  dimensional cross section on which the Killing field  $\xi^a$  vanishes, then  $\gamma_b^a \nabla_a \kappa = 0$  on the bifurcation surface and since the surface gravity can not change along the generator, this will establish the constancy of surface gravity on the entire Killing horizon. Hence, for Killing horizons with regular bifurcation surface, the surface gravity is constant irrespective of gravitational dynamics [18]. But the assumption of the existence of bifurcation surface is a strong assumption and it is only applicable to a sub-class of Killing horizons. We would like to know if as in the case of general relativity, the constancy of surface gravity of Killing horizons in Lanczos-Lovelock gravity can be established without these assumptions.

Let us now turn our attention to the features of Lanczos-Lovelock gravity. As discussed before, a natural generalization of the Einstein-Hilbert Lagrangian is provided by the Lanczos-Lovelock Lagrangian, which is the sum of dimensionally extended Euler densities,

$$\mathcal{L}^D = \sum_{m=0}^{[D-1]/2} \alpha_m \mathcal{L}_m^D, \quad (6)$$

where the  $\alpha_m$  are arbitrary constants and  $\mathcal{L}_m^D$  is the  $m$ -th order Lanczos-Lovelock term given by,

$$\mathcal{L}_m^D = \frac{1}{16\pi} \sum_{m=0}^{[D-1]/2} \frac{1}{2^m} \delta_{c_1 d_1 \dots c_m d_m}^{a_1 b_1 \dots a_m b_m} R_{a_1 b_1}^{c_1 d_1} \dots R_{a_m b_m}^{c_m d_m}, \quad (7)$$

where  $R_{ab}^{cd}$  is the  $D$  dimensional curvature tensor and the generalized alternating tensor  $\delta_{\dots}$  is totally antisymmetric in both set of indices. For  $D = 2m$ ,  $16\pi\mathcal{L}_m^D$  is the Euler density of  $2m$ -dimensional manifold. The Einstein-Hilbert Lagrangian is a special case of Eq. (7) when  $m = 1$ . The field equation of Lanczos-Lovelock theory is,  $G_{ab} + \alpha_m E_{(m)ab} = 8\pi T_{ab}$  where,

$$E_{(m)b}^a = -\frac{1}{2^{m+1}} \delta_{bc_1 d_1 \dots c_m d_m}^{aa_1 b_1 \dots a_m b_m} R_{a_1 b_1}^{c_1 d_1} \dots R_{a_m b_m}^{c_m d_m}, \quad (8)$$

and  $m \geq 2$ . For convenience, we have written the GR part (i.e. for  $m = 1$ ) separately so that the GR limit can be easily verified by setting all  $\alpha_m$ 's to zero.

Lanczos-Lovelock gravity can be regarded as a natural and well behaved extension of general relativity in higher dimensions. The spherically symmetric black hole solution in Lanczos-Lovelock gravity is derived in [19, 20]. The first law of black hole mechanics is studied in [21] and various thermodynamic properties of these black hole

solutions are discussed in [22].

Now, using the field equation of Lanczos-Lovelock gravity, we rewrite Eq.(5) as,

$$\gamma_b^a \nabla_a \kappa = -8\pi \xi^a T_{ac} \gamma_b^c + \alpha_m \xi_a E_{(m)c}^a \gamma_b^c. \quad (9)$$

Next, to simplify the above expression, we would first like to show that for a Killing horizon,  $E_{(m)b}^a \xi_a \xi^b = 0$ . In order to prove that, we first expand the curvature tensor in the basis  $\{\xi^a, N^a, e_A^a\}$  on the horizon. Therefore, we write,

$$R_{a_1 b_1}^{c_1 d_1} = g_p^{c_1} g_q^{d_1} g_{a_1}^r g_{b_1}^s R_{rs}^{pq}. \quad (10)$$

Now, as mentioned earlier, we express the space time metric as  $g_{ab} = -2\xi_{(a} N_{b)} + \gamma_{ab}$ . Also, stationarity ensures that some of the components are zero due to conditions Eq.(3) and Eq.(4). Then we can express,

$$\begin{aligned} R_{a_1 b_1}^{c_1 d_1} = & (\gamma_p^{c_1} \gamma_q^{d_1} \gamma_{a_1}^r \gamma_{b_1}^s - 2\gamma_p^{c_1} \gamma_{a_1}^r \gamma_{b_1}^s N_q \xi^{d_1} + 4\gamma_p^{c_1} \gamma_{a_1}^r \xi^{d_1} N_q \xi^s N_{b_1} + 2\gamma_{a_1}^r \gamma_{b_1}^s \xi^{c_1} N_p N^{d_1} \xi_q - 4\gamma_{a_1}^r \xi^{c_1} N_p N^{d_1} \xi_q \xi^s N_{b_1} \\ & - 2\gamma_p^{c_1} \gamma_q^{d_1} \gamma_{a_1}^r \xi_{b_1} N^s + 2\gamma_p^{c_1} \gamma_q^{d_1} \xi_{a_1} N^r \xi^s N_{b_1} + 4\gamma_p^{c_1} \gamma_{a_1}^r \xi^{d_1} \xi_{b_1} N_q N^s - 4\gamma_p^{c_1} \xi^{d_1} N_q \xi_{a_1} N^r \xi^s N_{b_1} \\ & + 4\gamma_p^{c_1} \gamma_{a_1}^r \xi_q N^{d_1} \xi_{b_1} N^s - 4\gamma_p^{c_1} \xi_q N^{d_1} \xi_{a_1} N^r \xi^s N_{b_1} - 4\gamma_{a_1}^r \xi^{c_1} N_p \xi_q N^{d_1} \xi_{b_1} N^s \\ & + 4\xi^{c_1} N_p \xi_q N^{d_1} \xi_{a_1} N^r \xi^s N_{b_1}) R_{rs}^{pq} \end{aligned} \quad (11)$$

We again remind the reader that this expression is valid only on the horizon. We also used the antisymmetry of the generalized alternating tensor  $\delta_{\dots}$ . Now any component of a curvature tensor along the direction of the Killing vector in the expression of  $E_{(m)b}^a$  will not contribute when contracted by  $\xi_a \xi^b$ . These constraints ensure that the only surviving contribution will be from the transverse components and as a result we get,

$$E_{(m)c}^a \xi_a \xi^c = -\frac{1}{2^{m+1}} \delta_{cC_1 D_1 \dots C_m D_m}^{aA_1 B_1 \dots A_{m-1} B_{m-1} A_m B_m} R_{A_1 B_1}^{C_1 D_1} \dots R_{A_m B_m}^{C_m D_m} \xi^c \xi_a, \quad (12)$$

where, indices  $(A_1, B_1, C_1, \dots)$  only take transverse values.

Next, we explicitly break the alternating tensor as the totally antisymmetric product of the Kronecker delta. For example, we write,

$$\begin{aligned} \delta_{c_1 d_1 \dots c_{m-1} d_{m-1} c_m d_m}^{a_1 b_1 \dots a_{m-1} b_{m-1} a_m b_m} &= \delta_c^a \delta_{c_1 d_1 \dots c_{m-1} d_{m-1} c_m d_m}^{a_1 b_1 \dots a_{m-1} b_{m-1} a_m b_m} \\ &- \delta_{c_1}^a \delta_{c d_1 \dots c_{m-1} d_{m-1} c_m d_m}^{a_1 b_1 \dots a_{m-1} b_{m-1} a_m b_m} + \delta_{d_1}^a \delta_{c c_1 \dots c_{m-1} d_{m-1} c_m d_m}^{a_1 b_1 \dots a_{m-1} b_{m-1} a_m b_m} \\ &+ \dots \end{aligned} \quad (13)$$

Note that, when contracted by  $\xi_a \xi^c$ , the first term vanishes and also the rest of the terms do not contribute if

all other indices are projected along the transverse directions.

Using this rule of expansion, we immediately see that on the horizon,  $E_{(m)b}^a \xi_a \xi^b = 0$ . Then, Eq.(3) and the field equation give that on the horizon  $T_b^a \xi_a \xi^b = 0$ .

Now, if the energy-momentum tensor obeys the dominant Energy Condition [12],  $T_b^a \xi^b$  will be a non-space like vector. But on the horizon, we have seen that the field equation implies  $T_b^a \xi_a \xi^b = 0$ . Therefore, to obey dominant energy condition  $T_b^a \xi^b$  must be in the direction of  $\xi^a$  only and as a result  $\xi^a T_{ac} \gamma_b^c = 0$ . So, we ultimately arrive at,

$$\gamma_b^a \nabla_a \kappa = \alpha_m \xi_a E_{(m)c}^a \gamma_b^c. \quad (14)$$

From the above equation, setting  $\alpha_m = 0$ , we can obtain the result of [6], which proves the constancy of surface gravity for GR.

We will now show that on the Killing horizon,  $\xi^a E_{(m)ac} \gamma_b^c$  does not vanish identically unless one imposes additional constraints on the geometry of the horizon.

To prove this, we again consider the expansion of the curvature tensor  $R_{a_1 b_1}^{c_1 d_1}$  in the basis  $\{\xi^a, N^a, e_A^a\}$  on the

horizon and use Eq.(3) and Eq.(4). Due to the antisymmetry of the generalized alternating tensor  $\delta_{\dots}$ , any component along  $\xi_{a_1}$  or  $\xi_{b_1}$  in the expression Eq.(11), will not contribute when contracted by  $\xi_a$ . Then, the only non

zero contributions of the expansion of  $R_{a_1 b_1}^{c_1 d_1}$  in the basis  $\{\xi^a, N^a, e_A^a\}$  are,

$$R_{A_1 B_1}^{C_1 D_1} - 2N_p R_{A_1 B_1}^{C_1 p} \xi^{d_1} + 4N_p \xi^q R_{A_1 q}^{C_1 p} \xi^{d_1} N_{b_1} + 2N_p \xi_q R_{A_1 B_1}^{pq} \xi^{c_1} N^{d_1} - 4N_p \xi_q \xi^r R_{A_1 r}^{pq} \xi^{c_1} N^{d_1} N_{b_1}. \quad (15)$$

We now consider products of two curvatures of the form  $R_{a_{m-1} b_{m-1}}^{c_{m-1} d_{m-1}} R_{a_m b_m}^{c_m d_m}$ . Due to the antisymmetry of the generalized alternating tensor  $\delta_{\dots}$ , the products of the com

ponents along the direction of the Killing vector will not contribute and the non vanishing contributions in the product  $R_{a_{m-1} b_{m-1}}^{c_{m-1} d_{m-1}} R_{a_m b_m}^{c_m d_m}$  can be expressed as,

$$R_{A_{m-1} B_{m-1}}^{C_{m-1} D_{m-1}} \left[ R_{A_m B_m}^{C_m D_m} - 4N_p R_{A_m B_m}^{C_m p} \xi^{d_m} + 8N_p \xi^q R_{A_m q}^{C_m p} \xi^{d_m} N_{b_m} + 4N_p \xi_q R_{A_m B_m}^{pq} \xi^{c_m} N^{d_m} - 8N_p \xi_q \xi^r R_{A_m r}^{pq} \xi^{c_m} N^{d_m} N_{b_m} \right] \quad (16)$$

Continuing in this way, we can express the product of

$m$ -curvature tensors appearing in  $\xi^a E_{(m)ac} \gamma_b^c$  as,

$$R_{A_1 B_1}^{C_1 D_1} \dots R_{A_{m-1} B_{m-1}}^{C_{m-1} D_{m-1}} \left[ R_{A_m B_m}^{C_m D_m} - 2^m \left( N_p R_{A_m B_m}^{C_m p} \xi^{d_m} - 2N_p \xi^q R_{A_m q}^{C_m p} \xi^{d_m} N_{b_m} - N_p \xi_q R_{A_m B_m}^{pq} \xi^{c_m} N^{d_m} + 2N_p \xi_q \xi^r R_{A_m r}^{pq} \xi^{c_m} N^{d_m} N_{b_m} \right) \right]. \quad (17)$$

This entire expression is contracted with the alternating tensor  $\xi^a \gamma_b^c \delta_{\dots}$ . Again, the expansion of the alternating tensor in Eq. (13) ensures that the first term in the above expansion is zero when contracted by  $\xi_a \gamma_b^c$ . Also, using

the expansion rule in Eq.(13), it is straightforward to see that the only non-vanishing contribution comes from the last term in Eq.(17), given by,

$$\begin{aligned} & \delta_{c_1 d_1 \dots c_{m-1} d_{m-1} c_m d_m}^{a_1 b_1 \dots a_{m-1} b_{m-1} a_m b_m} R_{A_1 B_1}^{C_1 D_1} \dots R_{A_{m-1} B_{m-1}}^{C_{m-1} D_{m-1}} R_{a_m b_m}^{c_m d_m} \xi_a \gamma_b^c \\ &= \xi_a \gamma_b^c \delta_{c_1 d_1 \dots c_{m-1} d_{m-1} c_m d_m}^{a_1 b_1 \dots a_{m-1} b_{m-1} a_m b_m} R_{A_1 B_1}^{C_1 D_1} \dots R_{A_{m-1} B_{m-1}}^{C_{m-1} D_{m-1}} R_{A_m r}^{pq} N_p \xi_q \xi^r \xi^{c_m} N^{d_m} N_{b_m}. \end{aligned} \quad (18)$$

Again, using the rules of expansion for the alternating tensor, it is straightforward to show that the only non

zero contribution from this term is of the form,

$$\begin{aligned} & \xi_a \gamma_b^c \delta_{d_m}^{c_m} \delta_{c_m}^{b_m} \delta_{c_1 d_1 \dots c_{m-1} d_{m-1}}^{A_1 B_1 \dots A_{m-1} B_{m-1} A_m} R_{A_1 B_1}^{C_1 D_1} \dots R_{A_{m-1} B_{m-1}}^{C_{m-1} D_{m-1}} R_{A_m r}^{pq} N_p \xi_q \xi^r \xi^{c_m} N^{d_m} N_{b_m} \\ &= 2^{m+1} \delta_{B C_1 D_1 \dots C_{m-1} D_{m-1}}^{A_1 B_1 \dots A_{m-1} B_{m-1} A_m} R_{A_1 B_1}^{C_1 D_1} \dots R_{A_{m-1} B_{m-1}}^{C_{m-1} D_{m-1}} R_{A_m r}^{pq} N_p \xi_q \xi^r. \end{aligned} \quad (19)$$

Since, for stationary black holes, both the expansion

and shear vanish, we can write [12],  $R_{AB}^{CD} = {}^{(D-2)} R_{AB}^{CD}$ ,

where,  ${}^{(D-2)}R_{AB}^{CD}$  is the intrinsic curvature of the cross section of the horizon. Then, we recall the expression for the equation of motion of a  $(m-1)$ -th order Lanczos-

Lovelock theory constructed using intrinsic curvatures of the horizon cross section, which is given by,

$${}^{(D-2)}E_{(m-1)b}^a = -\frac{1}{2^m} \delta_{b c_1 d_1 \dots c_{m-1} d_{m-1}}^{a a_1 b_1 \dots a_{m-1} b_{m-1}} {}^{(D-2)}R_{a_1 b_1}^{c_1 d_1} \dots {}^{(D-2)}R_{a_{m-1} b_{m-1}}^{c_{m-1} d_{m-1}}. \quad (20)$$

Using this expression, we finally arrive at,

$$(2^{-m}) \gamma_b^a \nabla_a \kappa = -\alpha_m {}^{(D-2)}E_{(m-1)b}^a R_{a r}^{p q} N_p \xi_q \xi^r, \quad (21)$$

Eq. (21) is our final expression and the right hand side of this equation does not vanish in general. So, unlike general relativity, for any higher order Lanczos-Lovelock theory of gravity, the surface gravity may vary from one generator to another on the horizon. As a result, although the surface gravity is constant along a single generator (i.e. surface gravity is independent of the affine parameter  $\lambda$  of the horizon), it may depend on the angular coordinates and vary as one moves across the generators.

In case of general relativity, the constancy of surface gravity for Killing horizons can be proved without any other assumption. Then, the rigidity theorems [14, 15] ensure that every stationary event horizon in GR must be a Killing horizon and this in turn, proves the constancy of the surface gravity for stationary black holes. Our work shows that higher order Lanczos-Lovelock terms do not share this property and if there exists a stationary solution of Lanczos-Lovelock gravity with Killing horizon, which is not axisymmetric with  $t - \phi$  symmetry, the surface gravity in general will be a function of the angular coordinates on a cross section of the horizon.

Let us now consider the special cases. First of all, if the cross section of the horizon is flat, i.e when the horizon topology is planar, all the intrinsic curvature tensors of the horizon cross section vanish and as a result, the surface gravity will not vary from one generator to another.

Also, if  $R_{br}^{pq} N_p \xi_q \xi^r = 0$  on the horizon, the surface gravity is again constant. A possible way to achieve this is to consider a stationary axisymmetric horizon with  $t - \phi$  isometry. For example, using equation 2.18 and 2.27 in [17], we can show that if the expansion and shear vanish, as in the case of a stationary black hole, then we have,  $R_{Br}^{pq} N_p \xi_q \xi^r = \xi^a R_{ac} \gamma_b^c$ , for a stationary horizon. Now, a sufficient condition which ensures that  $\xi^a R_{ac} \gamma_b^c = 0$ , is the existence of a stationary and axisymmetric horizon with  $t - \phi$  isometry [12]. This is basically the result obtained in [13].

Also, so far, we do not have any stationary solution of Lanczos-Lovelock gravity except general relativity. But, at least for general relativity with Gauss Bonnet correction terms, we have explicit spherically symmetric solutions [19, 20]. Given the quasi-linearity of the field equations of Lanczos-Lovelock gravity, it is quite possible that in this theory, a stationary solution will be found. If such solutions exist, our work shows that these black hole solutions will have non-constant surface gravity unless they are axisymmetric with  $t - \phi$  symmetry. Also, once we consider quantum effects, the surface gravity is proportional to the Hawking temperature of the black hole and hence, if the surface gravity is no longer constant on the horizon and varies from one generator to another, then we can not treat such a stationary black hole as a system in thermodynamic equilibrium.

But, there is still a possibility that as in the case of general relativity, all stationary horizons in a Lanczos-Lovelock theory are axisymmetric with  $t - \phi$  isometry. If that happens, then the zeroth law will be valid automatically. In order to investigate this, one needs to try for a generalization of the strong rigidity theorem [14, 15] for Lanczos-Lovelock gravity. The proof of the rigidity theorem depends on the initial value formulation of Einstein's equations. Since, the field equations of Lanczos-Lovelock gravity are also second order in time, and as a result the initial value formalism is well defined, it is reasonable to expect the validity of rigidity theorem for Lanczos-Lovelock gravity.

Another obvious generalization of our work will be to study the zeroth law of Killing horizons in a general diffeomorphism invariant theory of gravity. Although, the techniques used in this work are to some extent specific to only Lanczos-Lovelock gravity, still we can provide some requirements which will ensure the validity of the zeroth law for any diffeomorphism invariant theory. To begin with, let us consider a general diffeomorphism invariant theory of gravity described by an Lagrangian  $\mathcal{L}$ . Suppose, the field equation of the theory is given by  $\mathcal{E}_{ab} = 8\pi T_{ab}$ , where  $\mathcal{E}_{ab}$  represents a covariantly conserved, symmetric tensor obtained from the variation of the Lagrangian  $\mathcal{L}$ . Then, from Eq.(5), we obtain,

$$\begin{aligned} \gamma_b^a \nabla_a \kappa &= -\xi^a R_{ac} \gamma_b^c \\ &= \xi^a (\mathcal{E}_{ac} - R_{ac}) \gamma_b^c - 8\pi \xi^a T_{ac} \gamma_b^c. \end{aligned} \quad (22)$$



The zeroth law will hold if on the Killing horizon, following constraints are satisfied,

$$\mathcal{E}_{ab}\xi^a\xi^b = 0 \quad \text{and} \quad (\mathcal{E}_c^a - R_c^a)\xi_a\gamma_b^c = 0 \quad (23)$$

In general, it is difficult to check these constraints for a general gravity theory, but if the above two conditions hold and the matter source obeys dominant energy then that will be enough to ensure the constancy of surface gravity on the Killing horizon of a stationary space time. In case of general relativity, both of these constraints are satisfied and the zeroth law holds true even for a general Killing horizon. For Lanczos-Lovelock theory, the first constraint holds, but the second one is not true in general and as a result, the surface gravity is no more constant on the horizon.

In fact, it is also quite possible that the zeroth law does

not hold for a general stationary black hole in some class of gravity theories. In that case, this may be useful as a criterion to select a sub class of diffeomorphism invariant actions as preferred theories where a consistent formulation of black hole thermodynamics is possible.

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